Construction of generalized Hadamard matrices 

\[ D(r^m(r + 1), r^m(r + 1); p) \]

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Abstract

In this paper, a recurrent method for constructing the generalized Hadamard matrices \( D(r^m(r + 1), r^m(r + 1); p) \), \( m \geq 1 \), has been given under the condition that both \( D(r, r; r) \) and \( D(r + 1, r + 1; p) \) are known. The particular structures of the Hadamard matrices \( D(r^m(r + 1), r^m(r + 1); p) \) (or, equivalently, the orthogonal arrays \( L_m(r+1,p(m(r+1))) \)) constructed in this paper are also interesting since the matrix images of subarrays of the corresponding orthogonal arrays have clear and simple forms which can be obtained easily. The property can be used to construct the other new difference matrices and orthogonal arrays by using the methods of orthogonal decompositions of projection matrices (Zhang, Lu, Pang, Statist. Sinica 9 (1999) 595). © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Difference matrices have been very useful in constructing orthogonal arrays. The term “difference matrix” has been coined by Jungnickel (1979), but such matrices have already been explicitly used by Bose and Bush (1952). Recent survey papers on difference matrices and their applications are due to de Launey (1986).

Using the notation for additive groups, a difference matrix having level \( p \) is a \( \lambda p \times m \) matrix with the entries from a finite Abelian group \( G \) of cardinality \( p \) such that the

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vector differences of any two columns of the array, say \( d_i - d_j \) if \( i \neq j \), contain every element of \( G \) exactly \( \lambda \) times. We will denote such an array by \( D(\lambda, p, m; p) \), although this notation suppresses the relevance of the group \( G \). In most of our examples, \( G \) will correspond to the additive group associated with a Galois field \( \text{GF}(p) \). The difference matrix \( D(\lambda, p, m; p) \) is called a \textit{generalized Hadamard matrix} if \( \lambda p = m \). In particular, \( D(\lambda 2, \lambda 2; 2) \) is the usual Hadamard matrix.

The construction of difference matrices has been paid considerable attention in the literature, in part through the construction of generalized Hadamard matrices. The first family of Hadamard matrices which has been considered already is \( D(\lambda p^u, \lambda p^u; p^v) \), \( u \geq v \), \( p \) a prime power, and \( \lambda \) an order of Hadamard matrix. A construction for case \( \lambda = 1 \) has already been given by Bose and Bush (1952) and uses the Galois fields \( \text{GF}(p^u) \) and \( \text{GF}(p^v) \). Masuyama (1969a,b) presents the generalized Hadamard matrices of \( D(2p^u, 2p^u; p^v) \), \( u \geq v \), \( p \) an odd prime. Some slightly simpler forms for this family are given by Liu (1977), Xu (1979) and Jiang (1979). Although the latter authors get their results only for \( v = 1 \), they can be easily extended to any positive integer \( v \), which was observed by Xiang (1983). Furthermore, Dawson (1985) has constructed the generalized Hadamard matrices for case \( \lambda = 4 \). The particular forms of \( D(20, 20; 5) \) and \( D(12, 12; 3) \) also have been given by Zhang (1989, 1990b), Seiden (1954) and Zhang (1990a) independently. de Launey and Dawson (1992) have constructed the generalized Hadamard matrices for cases \( \lambda = 8, 19 < p < 200 \) or \( p > 19 \), \( p \) a prime power. Also for case \( \lambda = 8 \) the form of \( D(24, 24; 3) \), i.e., \( p = 3 \), has been given by Zhang (1991a) but only the form of \( D(24, 20; 4) \), i.e., \( p = 4 \), is exhibited in Zhang (1993) with its proof given in Zhang et al. (2000). Finally de Launey and Dawson (1994) have constructed the generalized Hadamard matrices for case \( \lambda = \) the order of Hadamard matrix and the prime power \( p > (\lambda - 2)^2 \).

The second family of Hadamard matrices that has also been considered already is \( D(r^m(r + 1), r^m(r + 1); p) \), \( m \geq 1 \), where both \( r \) and \( p \) are prime powers and \( D(r + 1, r + 1; p) \) is existent. Although some particular forms of this family, such as \( D(30, 30; 3) \), \( D(20, 20; 5) \) and \( D(24, 24; 3) \), have been given by many scholars, such as Masuyama (1957), Street (1979a, b), Zhang (1990b, 1991a), the generalized construction of the family has not yet received much attention in the literature since the problem is too difficult to solve. Warwick de Launey (1986) announced the recursive construction for an elementary Abelian group and gave an example for \( m = 1 \). Note that the special case \( p = r + 1 \) includes the result of Seberry (1980). For case \( m = 2 \), see also de Launey (1989). The result is proved in full in his Ph.D. Thesis (de Launey, 1988). Unfortunately when Warwick de Launey sent his manuscript to many journals such as Discrete Mathematics in the mid 1980s, after some time he was told that they could not find anyone who was willing to referee it because his proof is too difficult even for the elementary Abelian group, so he let it drop. But a proof for this family over any finite group \( G \) has been obtained by Zhang (1993) independently. The proof follows from a new theory — the theory of multilateral matrix which is a mathematical technique to solve the problems of system with complexity. In this paper, a simple proof will be given. In fact, we will prove that the generalized Hadamard matrix
Let $D(r^m(r + 1), r^m(r + 1); p)$ be constructed if both $D(r, r; r)$ and $D(r + 1, r + 1; p)$ are known.

In order to prove our results, we need to consider the relationship between difference matrices and orthogonal arrays.

If a $D(\lambda p, m; p)$ exists, it can always be constructed so that only one of its rows and one of its columns has the zero element of $G$. Deleting this column from $D(\lambda p, m; p)$, we obtain a difference scheme, denoted by $D^0(\lambda p, m – 1; p)$, called an atom of the difference matrix $D(\lambda p, m; p)$. Without loss of generality, $D(\lambda p, m; p)$ can be written as

$$D(\lambda p, m; p) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = [0 \quad D^0(\lambda p, m – 1; p)].$$

This property is important for the following discussions.

For two matrices $A = (a_{ij})_{n \times m}$ and $B = (b_{ij})_{s \times t}$ both with entries from the group $G$, define their Kronecker sum (Shrikhande, 1964) as

$$A \oplus B = (a_{ij} + B)_{1 \leq i \leq n, 1 \leq j \leq m},$$

where each submatrix $a_{ij} + B$ of $A \oplus B$ stands for the matrix obtained by adding $a_{ij}$ to each entry of Shrikhande (1964) showed that $A \oplus B$ is a difference matrix if both $A$ and $B$ are difference matrices. By contrast, Zhang (1993) showed that $A$ is a different matrix if both $A \oplus B$ and $B$ are difference matrices.

It is known that the Kronecker sum

$$L = L_{\mu p}(p^s) \oplus D(\lambda p, m; p) \quad \text{(or } L = D(\lambda p, m; p) \oplus L_{\mu p}(p^s))$$

is an orthogonal array if $L_{\mu p}(p^s)$ is an orthogonal array and $D(\lambda p, r; p)$ is a difference matrix (Beth et al., 1986). If $\mu = s = 1$ the Kronecker sum method reduces to the well-known construction of Bose and Bush (1952), i.e., the following:

$$L = (p) \oplus D(\lambda p, m; p) \quad \text{(or } L = D(\lambda p, m; p) \oplus (p))$$

is an orthogonal array if $D(\lambda p, m; p)$ is a difference matrix. By contrast, Zhang (1993) has found that the difference matrix $D(\lambda p, m; p)$ can also be constructed by using the orthogonal array $L = (p) \oplus D(\lambda p, m; p)$, i.e., $D(\lambda p, m; p)$ is a difference matrix if $L = (p) \oplus D(\lambda p, m; p)$ is an orthogonal array. In this paper, the idea will be used to construct the generalized Hadamard matrix $D(r^m(r + 1), r^m(r + 1); p)$ by using the construction of orthogonal array $L_{r^m(r + 1)p}(p^{r^m(r + 1)}) = (p) \oplus D(r^m(r + 1), r^m(r + 1); p)$.

In Section 2, we present the method for the construction of $D(r(r + 1), r(r + 1); p)$ from $D(r, r; r)$ and $D(r + 1, r + 1; p)$. In Section 3, we prove a recurrent method for constructing the particular orthogonal array $L_{r^m(r + 1)p}(p^{r^m(r + 1)}) = (p) \oplus D(r^m(r + 1), r^m(r + 1); p)$. As the main result of this paper, the procedure for the construction of $D(r^m(r + 1), r^m(r + 1); p)$ is presented in Section 4. We also elaborate on the construction steps in Section 5 by some examples. A particular structure of the orthogonal array $L_{18}(6^{136})$ will be given in Example 5.1, which can be used to construct the other new orthogonal arrays and difference matrices.
2. Construction of $D(r(r+1), r(r+1); p)$

In our procedure, an important idea is to find the relationship among difference matrices, permutation matrices and projection matrices. The following notations are denoted as follows:

Let $1_r$ be the $r \times 1$ vector of 1’s, 0, be the $r \times 1$ vector of 0’s, $I_r$ the identity matrix of order $r$ and $J_{r,s}$ the $r \times s$ matrix of 1’s, also $J_r =: J_{r,r}$. Of course, the two matrices $P_{r} = (1/r)1_r, 1_r^T = (1/r)J_r$ and $\tau_r = I_r - P_r$ are projection matrices.

Define

\[
(r) = (0, \ldots, r - 1)^T_{1 \times r}, \quad e_i(r) = (0 \cdots 01 \cdots 0)^T_{1 \times r},
\]

where $(*^T$ means the transpose of matrix * and $e_i(r)$ is the base vector of $R^r$ ($r$-dim vector space) for any $i$. By the $r \times 1$ base vector $e_i(r)$, we can construct two $r \times r$ permutation matrices as follows:

\[
N_r = e_1(r)e_2^T(r) + \cdots + e_{r-1}(r)e_r^T(r) + e_r(r)e_1^T(r)
\]

and

\[
K(p, q) = \sum_{i=1}^{p} \sum_{j=1}^{q} e_i(p)e_j^T(q) \otimes e_j(q)e_i^T(p),
\]

where $\otimes$ is the usual Kronecker product in the theory of matrices. The permutation matrices $N_r, K(p, q)$ have the following properties:

\[
N_r \cdot (r) = 1_r + (r), \text{ mod } r \quad \text{and} \quad K(p, q) \cdot ((q) \otimes (p)) = (p) \otimes (q).
\]

Let $D(\lambda, p, m; p) = (d_{ij})_{i,j; p \times m}$ be a difference matrix over the finite group $G = \{0, 1, \ldots, p - 1\}$. Then for any given $d_{ij} \in G$, there exists a permutation matrix $\sigma(d_{ij})$ such that

\[
\sigma(d_{ij}) \cdot (p) = d_{ij} + (p).
\]

Define $H(\lambda, p, m; p) = (\sigma(d_{ij}))_{i,j; p \times m, p}$, where each entry $\sigma(d_{ij})$ of $H(\lambda, p, m; p)$ is the $p \times p$ permutation matrix. Then Zhang (1993) has proved that the matrix $D(\lambda, p, m; p) = (d_{ij})_{i,j; p \times m}$ over some group $G$ is a difference matrix if and only if

\[
H^T(\lambda, p, m; p)H(\lambda, p, m; p) = \lambda p(I_m \otimes \tau_p + J_m \otimes P_p).
\]

This is a useful relationship among the difference matrices, permutation matrices and projection matrices in our procedure.

On the other hand, the permutation matrices $\sigma(d_{ij})$ are often obtained by the permutation matrices $N_r$ and $K(p, q)$. For example, if $G$ is the cyclic group of $p$ elements, then we have $\sigma(d_{ij}) = N^d_{ij}$. Furthermore, by the permutation matrices $\sigma(d_{ij})$ and $K(\lambda, p, m)$, the Kronecker sum (Shrikhande, 1964) of difference matrices can be written as

\[
(p) \oplus D(\lambda, p, m; p) = K(p, \lambda p) \cdot D(\lambda, p, m; p) \oplus (p)
\]

\[
= K(p, \lambda p) \cdot (\sigma(d_{ij})(p))_{i,j; p^2 \times m}
\]

\[
= K(p, \lambda p) \cdot (S_1(0_{i,p} \oplus (p)), \ldots, S_m(0_{i,p} \oplus (p)))
\]

\[
= K(p, \lambda p) \cdot (Q_1((p) \oplus 0_{i,p}), \ldots, Q_m((p) \otimes 0_{i,p})),
\]
where
\[ Q_j = S_j K(p, \lambda p)^T, \quad S_j = \text{diag}(\sigma(d_{1j}), \ldots, \sigma(d_{rj})), \quad r = \lambda p \] (2)
is a permutation matrix for any \( j = 1, \ldots, m \) and \( 0_{\lambda p} \oplus (p) = 1_{\lambda p} \otimes (p) \) holds for the additive group associated with a Galois field \( GF(p) \). Therefore, the projection matrices \( P_r, \tau_r \) and the permutation matrices \( N_r, K(p, q) \) and \( S_j \) (defined in (2)) are often used to construct the asymmetrical orthogonal arrays in our procedure.

These properties are often used in the construction of generalized Hadamard matrix \( D(r+1, r+1, p) \).

**Theorem 2.1.** Suppose that both
\[ D(r, r; r) = (d_{ij})_{r \times r} = (d_1, \ldots, d_r) \] (3)
and
\[ D(r+1, r+1; p) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (a_{ij})_{r \times r} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & a_r \end{pmatrix} \] (4)
are generalized Hadamard matrices with entries from \( G_r = \{0, 1, \ldots, r - 1\} \) and \( G_p = \{0, 1, \ldots, p - 1\} \), respectively. For any \( d_{ij} \in G_r \), define a permutation matrix \( \sigma(d_{ij}) \) such that
\[ \sigma(d_{ij}) \cdot (r) = d_{ij} + (r). \] (5)

Let \( F = (\sigma(d_{ij})A)_{1 \leq i \leq r, 1 \leq j \leq r} \). Then the following array:
\[ D(r+1, r+1; p) = \begin{pmatrix} 0 & A \oplus 0_r^T \\ A \oplus 0_r & F \end{pmatrix}, \] (6)
is a generalized Hadamard matrix.

**Proof.** (a) Consider the following submatrix of \( D(r+1, r+1; p) \):
\[ B = \begin{pmatrix} 0 & a_j \oplus 0_r^T \\ \sigma(d_{1j})A \\ A \oplus 0_r & \cdots \\ \sigma(d_{ij})A \end{pmatrix}. \] (7)
Let \( T = \text{diag}(I_r, \sigma^{-1}(d_{1j}), \ldots, \sigma^{-1}(d_{ij})) \). Then we have
\[ B' = TB = \begin{pmatrix} a_0 \oplus 0_r^T & a_j \oplus 0_r^T \\ A \oplus 0_r & 0_r \oplus A \end{pmatrix}, \]
where \( a_0 \) is \( r \times 1 \) vector of 0’s. From (4), both
\[ \begin{pmatrix} a_0 \oplus 0_r^T \\ A \oplus 0_r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_j \oplus 0_r^T \\ 0_r \oplus A \end{pmatrix} \]
are difference matrices. Now we consider the difference vector of the $k$th column of 
\[
\left(\begin{array}{c}
a_0 \\
0 \\
a_k \\
\vdots \\
a_k \\
\end{array}\right)
\] and the $l$th column of 
\[
\left(\begin{array}{c}
a_1 \\
0 \\
a_1 \\
\vdots \\
a_1 \\
\end{array}\right),
\]
i.e., 
\[
\left(\begin{array}{c}
a_j - a_0 \\
a_l - a_1k \\
\vdots \\
a_l - a_1k \\
\end{array}\right).
\]

By (4), there exists a permutation matrix $S$, such that 
\[
S \left(\begin{array}{c}
a_j - a_0 \\
0 \\
0 \\
\vdots \\
0 \\
\end{array}\right) = 
\left(\begin{array}{c}
0 \\
0 \\
a_k \\
\vdots \\
a_k \\
\end{array}\right).
\]

Let $T = \text{diag}(S, I_r)$, then 
\[
T \cdot 
\left(\begin{array}{c}
a_j - a_0 \\
a_l - a_1k \\
\vdots \\
a_l - a_1k \\
\end{array}\right) = 
\left(\begin{array}{c}
0 \\
0 \\
a_k \\
\vdots \\
a_k \\
\end{array}\right) = 0_r \oplus 0_r + 1 \oplus a_l - (0 \oplus 0_r).
\]

Consider 
\[
0_r \otimes 
\left(\begin{array}{c}
0 \\
0 \\
a_k \\
\vdots \\
a_k \\
\end{array}\right) = 
\left(\begin{array}{c}
0 \\
0 \\
a_k \\
\vdots \\
a_k \\
\end{array}\right) - a_l \oplus 0_r + 1 = 
\left(\begin{array}{c}
0 \\
0 \\
a_k \\
\vdots \\
a_k \\
\end{array}\right).
\]

From (4), each element of $G$ in (8) occurs $r(r + 1)/p$ times. Therefore (7) is a 
difference matrix.

(b) Consider the following submatrix of $D(r(r + 1), r(r + 1); p)$:

\[
B = 
\left(\begin{array}{cc}
a_i \otimes 0_r^T & a_j \otimes 0_r^T \\
\sigma(d_{ij})A & \sigma(d_{ij})A \\
\sigma(d_{ij})A & \sigma(d_{ij})A \\
\end{array}\right),
\quad (i \neq j).
\]

By (3) and the construction of Bose and Bush (1952), 
$L = D(r, r; r) \oplus (r) = (\sigma(d_{ij})(r))_{r^2 \times r}$ is an orthogonal array, so that the following subarray of $L$:

\[
\left(\begin{array}{cc}
\sigma(d_{ij}) \cdot (r) & \sigma(d_{ij}) \cdot (r) \\
\sigma(d_{ij}) \cdot (r) & \sigma(d_{ij}) \cdot (r) \\
\sigma(d_{ij}) \cdot (r) & \sigma(d_{ij}) \cdot (r) \\
\end{array}\right) = (d_i, d_j) \oplus (r),
\quad (i \neq j)
\]

is also an orthogonal array. By the definition of orthogonal array and (10), there exists a permutation matrix $T$ such that 
\[
T \cdot 
\left(\begin{array}{cc}
\sigma(d_{ij}) \cdot (r) & \sigma(d_{ij}) \cdot (r) \\
\sigma(d_{ij}) \cdot (r) & \sigma(d_{ij}) \cdot (r) \\
\sigma(d_{ij}) \cdot (r) & \sigma(d_{ij}) \cdot (r) \\
\end{array}\right) = ((r) \oplus (r), (r) \oplus (r)),
\quad (i \neq j).
Let $T' = \text{diag}(I_r, T)$. Then we have

$$B' = T'B = \left( \begin{array}{cc}
    a_i \oplus 0_r^T & a_j \oplus 0_r^T \\
    A \oplus 0_r & 0_r \oplus A
\end{array} \right).$$

The $B'$ is a difference matrix according to (a). Therefore (9) is a difference matrix. The proof is completed. □

3. Construction of \(\text{OA } L_{r^m (r+1)p} (p^{r^m (r+1)})\)

In order to construct the particular orthogonal array \(L_{r^m (r+1)p} (p^{r^m (r+1)}) = (p) \oplus D(r^m(r+1), r^{m(r+1)}; p)\), we also need to know the relationship between orthogonal arrays and projection matrices. A new concept of the so-called matrix image is given in the following.

**Definition 3.1.** Let $A$ be an orthogonal array of strength 1, i.e.

$$A = (a_1, \ldots, a_m) = \left[ S_1(0_{r_1} \oplus (p_1)), \ldots, S_m(0_{r_m} \oplus (p_m)) \right],$$

where $r_i p_i = n$, $S_i$ is a permutation matrix for any $i = 1, \ldots, m$. The following projection matrix:

$$A_j = S_j(P_{r_j} \otimes \tau_{p_j}) S_j^T \quad (11)$$

is called the matrix image (MI) of the $j$th column $a_j$ of $A$, denoted by $m(a_j) = A_j$ for $j = 1, \ldots, m$. In general, the MI of a subarray of $A$ is defined as the sum of the MI of all its columns. In particular, we denote the MI of $A$ by $m(A)$.

If a design is an orthogonal array, then the matrix images of its columns have some interesting properties which can be used to construct orthogonal arrays. For example, by the definition, we have

$$m(0_r) = P_r \quad \text{and} \quad m((r)) = \tau_r.$$ 

In general, we have the following two elementary Theorems.

**Theorem 3.1.** For any permutation matrix $S$ and any array $L$:

$$m(S(L \oplus 0_r)) = S(m(L) \otimes P_r) S^T \quad \text{and} \quad m(S(0_r \oplus L)) = S(P_r \otimes m(L)) S^T.$$ 

**Theorem 3.2.** Let the array $A$ be an orthogonal array of strength 1, i.e.

$$A = (a_1, \ldots, a_m) = \left[ S_1(0_{r_1} \oplus (p_1)), \ldots, S_m(0_{r_m} \oplus (p_m)) \right],$$

where $r_i p_i = n$, $S_i$ is a permutation matrix, for $i = 1, \ldots, m$.

The following statements are equivalent:

1. $A$ is an orthogonal array of strength 2.
2. The MI of $A$ is a projection matrix.
(3) The MI of any two columns of \( A \) are orthogonal, i.e., \( m(a_i)m(a_j) = 0 (i \neq j) \).

(4) The projection matrix \( \tau_n \) can be decomposed as

\[
\tau_n = m(a_1) + \cdots + m(a_m) + \Delta,
\]

where \( \text{rk}(\Delta) = n - 1 - \sum_{j=1}^{m} (p_j - 1) \) is the rank of the matrix \( \Delta \).

**Definition 3.2.** An orthogonal array \( A \) is said to be saturated if \( \sum_{j=1}^{m} (p_j - 1) = n - 1 \) (or, equivalently, \( m(A) = \tau_n \)).

**Corollary 3.1.** Let \((L,H)\) and \(K\) be orthogonal arrays of row size \( n \). Then \((K,H)\) is an orthogonal array if \( m(K) \leq m(L) \); where \( m(K) \leq m(L) \) means that the difference \( m(K) - m(L) \) is nonnegative definite.

**Corollary 3.2.** Suppose \( L \) and \( H \) are orthogonal arrays. Then \( K = (L,H) \) is also an orthogonal array if \( m(L) \) and \( m(H) \) are orthogonal, i.e., \( m(L)m(H) = 0 \). In this case \( m(K) = m(L) + m(H) \).

These theorems and corollaries can be found in Zhang (1991b, 1992, 1993). Our procedure of constructing mixed-level orthogonal arrays by using orthogonal decompositions of projection matrix \( \tau_n \) consists of the following three steps (Zhang et al., 1999), also namely orthogonal-array addition:

**Step 1:** Orthogonally decompose the projection matrix \( \tau_n \): \( \tau_n = A_1 + \cdots + A_k \), where \( A_iA_j = 0 (i \neq j) \).

**Step 2:** Find an orthogonal array \( L_i \) such that \( m(L_i) \leq A_i \).

**Step 3:** Lay out the new orthogonal array \( L \) by Corollaries 3.1 and 3.2

\[
L = (L_1, \ldots, L_{k_1}), \quad (k_1 \leq k).
\]

The following theorem plays a very useful role in the construction of \( L = (p) \oplus D(r^m(r+1), r^m(r+1); p) \).

**Theorem 3.3.** For \( m = 0,1 \) and \( 1 \leq j \leq r \), assume that there exist permutation matrices \( Q_j \)'s, \((T_j^m)'s and projection matrices \((A_j^m)'s such that the following two equations of orthogonal decompositions of projection matrices hold.

\[
\tau_{r^2} = \sum_{j=0}^{r} Q_j \cdot \tau_r \otimes P_r \cdot Q_j^T \quad (12)
\]

and

\[
\tau_p \otimes I_{r^m(r+1)} = A_0^m + A_1^m + \cdots + A_r^m,
\]

which satisfy the following conditions (a)–(c):

(a) \( A_0^1 = (A_0^0 + \cdots + A_r^0) \otimes P_r \).

(b) \( A_j^m = T_j^m \cdot A_0^m \cdot (T_j^m)^T \).

(c) \( \tau_p \otimes I_{r^m(r+1)} = T_j^m \cdot \tau_p \otimes I_{r^m(r+1)} \cdot (T_j^m)^T \).
Then for $m \geq 2$ and $1 \leq j \leq r$, there also exist permutation matrices $(T^m_j)'s = (T^{m-2}_j \otimes Q_j)'s$ and projection matrices $(A^m_j)'s$ such that the following equations of orthogonal decompositions of projection matrix $\tau_p \otimes I_{m(r+1)}$ also hold:

$$
\tau_p \otimes I_{m(r+1)} = A^m_0 + A^m_1 + \cdots + A^m_r, \quad m \geq 2,
$$

which also satisfy the following conditions (a)–(c):

(a) $A^m_0 = (A^{m-1}_1 \otimes \cdots \otimes A^{m-1}_r) \otimes P_r$.

(b) $A^m_0 = T^m_j \cdot A^m_0 \cdot (T^m_j)^T$.

(c) $\tau_p \otimes I_{m(r+1)} = T^m_j \cdot \tau_p \otimes I_{m(r+1)} \cdot (T^m_j)^T$.

**Proof.** The theorem is proved by the recurrent method. When $m = 0, 1$ the proposition is right by the supposition. Now let us prove the proposition for $m \geq 2$ is right if it is right for $m - 1$, $m - 2$.

(a) By the recurrent supposition and using properties $I_{m-1(r+1)} = I_{m-2(r+1)} \otimes I_r$ and $I_r = \tau_r + P_r$, we obtain

$$
A^{m-1}_0 + A^{m-1}_1 + \cdots + A^{m-1}_r = \tau_p \otimes I_{m-1(r+1)} = \tau_p \otimes I_{m-2(r+1)} \otimes \tau_r + \tau_p \otimes I_{m-2(r+1)} \otimes P_r

= \tau_p \otimes I_{m-2(r+1)} \otimes \tau_r + (A^{m-2}_0 + A^{m-2}_1 + \cdots + A^{m-2}_r) \otimes P_r

= \tau_p \otimes I_{m-2(r+1)} \otimes \tau_r + A^{m-2}_0 \otimes P_r + A^{m-1}_0.
$$

Therefore, if we define

$$
A^m_0 := \tau_p \otimes I_{m-2(r+1)} \otimes \tau_r \otimes P_r + A^{m-2}_0 \otimes P_r = (A^{m-1}_1 + \cdots + A^{m-1}_r) \otimes P_r
$$

then we have

$$
A^m_0 = (\tau_p \otimes I_{m-2(r+1)} \otimes \tau_r \otimes P_r + A^{m-2}_0 \otimes P_r) \otimes P_r. \quad (14)
$$

Of course, condition (a) is satisfied.

(b) By the recurrent supposition, (12)–(14), and by using properties $(ABC) \otimes (DEF) = (A \otimes D)(B \otimes E)(C \otimes F)$ and $I_{r+2} = \tau_{r+2} + P_{r+2}$, we have

$$
\tau_p \otimes I_{m(r+1)} = \tau_p \otimes I_{m-2(r+1)} \otimes \tau_{r+2} + \tau_p \otimes I_{m-2(r+1)} \otimes P_{r+2}

= \tau_p \otimes I_{m-2(r+1)} \otimes \left( \sum_{j=0}^r Q_j \cdot \tau_r \otimes P_r \cdot Q_j^T \right)

+ \sum_{j=0}^r \left[ T^m_j \cdot A^m_0 \cdot (T^m_j)^T \right] \otimes P_{r+2}

= \sum_{j=0}^r (T^m_j \otimes Q_j) \cdot (\tau_p \otimes I_{m-2(r+1)} \otimes \tau_r

+ A^{m-2}_0 \otimes P_r) \otimes P_r \cdot (T^m_j \otimes Q_j)^T

= \sum_{j=0}^r (T^m_j \otimes Q_j) \cdot A^m_0 \cdot (T^m_j \otimes Q_j)^T.
$$
Therefore, if define $T_j^m = T_j^{m-2} \otimes Q_j$ and $A_j^m = T_j^m \cdot A_0^m \cdot (T_j^m)^T$, the desired equation of orthogonal decomposition of the projection matrix $\tau_p \otimes I_{r^{m(r+1)}}$ holds for any $m \geq 2$, i.e.,

$$\tau_p \otimes I_{r^{m(r+1)}} = \sum_{j=0}^{r} (T_j^m) A_0^m (T_j^m)^T = \sum_{j=0}^{r} A_j^m.$$  

Of course, condition (b) is also satisfied.

(c) According to the forms of both the permutation matrix $T_j^m = T_j^{m-2} \otimes Q_j$ (defined in Step (b)) and the projection matrix $\tau_p \otimes I_{r^{m(r+1)}} = \tau_p \otimes I_{r^{m-2(r+1)}} \otimes I_{r^2}$, and by the recurrent supposition, we have

$$\begin{align*}
(T_j^m) \cdot (\tau_p \otimes I_{r^{m(r+1)}}) \cdot (T_j^m)^T &= (T_j^{m-2} \otimes Q_j) \cdot (\tau_p \otimes I_{r^{m-2}} \otimes I_{r^2}) \cdot (T_j^{m-2} \otimes Q_j)^T \\
&= (T_j^{m-2} \cdot \tau_p \otimes I_{r^{m-2(r+1)}} \cdot (T_j^{m-2})^T) \otimes (Q_j I_{r^2} Q_j^T) \\
&= \tau_p \otimes I_{r^{m-2(r+1)}} \otimes I_{r^2} \\
&= \tau_p \otimes I_{r^{m(r+1)}},
\end{align*}$$

i.e., condition (c) is also satisfied for any $m \geq 2$. The proof is completed.

According to the relationship between the orthogonal arrays and orthogonal decompositions of projection matrices in Theorems 3.1 and 3.2, by using the procedure of three steps or the orthogonal-array addition, Theorem 3.3 can be written as follows:

**Theorem 3.4.** For $m = 0, 1$ and $1 \leq j \leq r$, assume that there exist permutation matrices $Q_j's$, $(T_j^m)'s$ and orthogonal arrays $(L_j^m)'s$ such that the following partitions hold:

$$L^0 = L_{(r+1)p} = (L_0^0, L_1^0, \ldots, L_r^0),$$

$$L^1 = L_{r(r+1)p} = (L_0^1, L_1^1, \ldots, L_r^1)$$

and

$$L_{r^2(r+1)} = [(r) \oplus 0_r, Q_1 \cdot (r) \oplus 0_r, \ldots, Q_r \cdot (r) \oplus 0_r],$$

which satisfy the following conditions:

(a) $L_0^1 = (L_1^0, \ldots, L_r^0) \oplus 0_r$.

(b) $L_j^m = T_j^m \cdot L_0^m \cdot (T_j^m)^T$.

(c) $\tau_p \otimes I_{r^{m(r+1)}} = T_j^m \cdot \tau_p \otimes I_{r^{m(r+1)}} \cdot (T_j^m)^T$.

Then for $m \geq 2$ and $1 \leq j \leq r$, there also exist permutation matrices $(T_j^m)'s = (T_j^{m-2} \otimes Q_j)'s$ and orthogonal arrays $(L_j^m)'s$ such that the following partitions also hold:

$$L^m = L_{r^{m(r+1)}p} = (L_0^m, L_1^m, \ldots, L_r^m), \quad m \geq 2,$$
which also satisfy the following conditions:

(a) \( L_0^m = (L_1^{m-1}, \ldots, L_r^{m-1}) \oplus 0_r. \)
(b) \( L_j^m = T_j^m \cdot L_0^m \cdot (T_j^m)^\top. \)
(c) \( \tau_p \otimes I_{r^m(r+1)} = T_j^m \cdot \tau_p \otimes I_{r^m(r+1)} \cdot (T_j^m)^\top. \)

3.1. The discussion of the existence of the supposition of Theorems 3.3 and 3.4

Suppose that there exists a generalized Hadamard matrix \( D(r+1, r+1; p) \) with \( r \) a prime power. Then the construction of \( D(r, r; r) \) (or \( L_r (r^{r+1}) \)) can be known (Bose and Bush, 1952). By Theorem 2.1, \( D(r+1, r(r+1); p) \) (or \( L_{r+1}(p+1) = [0_p \oplus ([r(r+1)])], (p) \oplus D(r+1, r(r+1); p)) \) can also be constructed.

In order to construct the difference matrices \( D(r^m(r+1, r^m(r+1); p), \) by Theorem 3.4, we need to know the structure of the particular orthogonal array \( L^m = (L_0^m, L_1^m, \ldots, L_r^m) = L_{r^m(r+1); p}(r^m(r+1)) \), i.e., to show \( L^m = (p) \oplus D(r^m(r+1), r^m(r+1); p). \)

In fact, the orthogonal array

\[
L_{(r+1)p} = [0_p \oplus (r + 1), (p) \oplus D(r + 1, r + 1; p)]
\]

is saturated since \((r + 1)p - 1 = (r + 1 - 1) + (r + 1)(p - 1).\) So by Theorem 3.2, we have \( m(L_{(r+1)p}) = \tau_{(r+1)p}. \) Thus by Corollary 3.2, we get

\[
m((p) \oplus D(r + 1, r + 1; p)) = \tau_{(r+1)p} - P_p \otimes \tau_{r+1} = \tau_p \otimes I_{r+1}.
\]

By (2), the orthogonal array \((p) \oplus D(r + 1, r + 1; p)\) can be written as

\[
L^0 = (p) \oplus D(r + 1, r + 1; p) = (L_0^0, L_1^0, \ldots, L_r^0),
\]

where \( L_0^0 = (p) \oplus 0_{r+1}, L_j^0 = (p) \oplus (0 a_j^T)^\top, (1 \leq j \leq r). \) The permutation matrix \( T_j^0 \) in Theorems 3.3 and 3.4 can be obtained from \( L_j^0 = T_j^0 L_0^0 \) for any \( j = 1, \ldots, r, \) i.e.

\[
T_j^0 = K(p, r + 1) \cdot \text{diag}(I_r, \sigma(a_1), \ldots, \sigma(a_r)) \cdot K(p, r + 1)^\top.
\]

Similarly, by the construction of \( D(r+1, r(r+1); p) \) in Theorem 2.1 and the construction of Bose and Bush (1952)

\[
L_{r^2}(r^{r+1}) = [(r) \oplus 0_r, D(r, r; r) \oplus (r)] = [(r) \oplus 0_r, Q_1((r) \oplus 0_r), \ldots, Q_r((r) \oplus 0_r)]
\]

we have

\[
L^1 = (p) \oplus D(r+1, r(r+1); p) = (L_0^1, L_1^1, \ldots, L_r^1),
\]

where

\[
L_0^1 = (p) \oplus \begin{pmatrix} 0 \\ A \end{pmatrix} \oplus 0_r = (L_1^0, \ldots, L_r^0) \oplus 0_r,
\]

\[
L_j^1 = (p) \oplus \begin{pmatrix} a_j \oplus 0_r^T \\ Q_j(\dot{A} \oplus 0_r) \end{pmatrix}, \quad (1 \leq j \leq r).
\]
And the permutation matrix $T_j^1$ in Theorems 3.3 and 3.4 can be obtained from $L_j^1 = T_j^1 L_0^1$ for any $j = 1, \ldots, r$, i.e.

$$T_j^1 = K(p, r(r + 1)) \cdot \text{diag}(\sigma(a_{ij}), \ldots, \sigma(a_{ij}), Q_j \otimes I_p) \cdot K(p, r(r + 1))^T.$$

From the recurrent construction in Theorem 3.3, the orthogonal array $L_{r^m(r+1)p}$ can be written as $L^m = (p) \oplus D(r^m(r+1), r^m(r+1); p)$.

The generalized Hadamard matrices $D(r^m(r+1), r^m(r+1); p)$ can be constructed by the construction of $L_{r^m(r+1)p} = L^m = (p) \oplus D(r^m(r+1), r^m(r+1); p)$ (Zhang, 1993). But the construction of $D(r^m(r+1), r^m(r+1); p)$ is complicated yet and it is difficult to get it directly. In the Section 4 the concrete steps are presented.

4. A general method for constructing difference matrices $D(r^m(r+1), r^m(r+1); p)$

**Definition 4.1.** Let $D(\lambda, p, r; p)$ be a difference matrix. The following transformations and called difference transformations:

1. To exchange any two rows of $D(\lambda, p, m; p)$.
2. To exchange any two columns of $D(\lambda, p, m; p)$.
3. To add a constant $c$ into some row or column of $D(\lambda, p, m; p)$.

By the definition of difference matrix, the $D(\lambda, p, m; p)$ on which any difference transformation is done is still a difference matrix.

Our procedure of constructing $D(r^m(r+1), r^m(r+1); p)$ by using Theorem 3.4 consists of the following four steps:

**Step 1:** Construct $D(r(r+1), r(r+1); p)$ by using $D(r+1, r+1; p)$ and $D(r, r; r)$. Let

$$D(r+1, r+1; p) = (H_0^0, H_1^0, \ldots, H_r^0).$$

Using Theorem 2.1, the difference matrix $D(r(r+1), r(r+1); p)$ can be constructed and can be written as

$$D(r(r+1), r(r+1); p) = (H_0^1, H_1^1, \ldots, H_r^1) = (H_0^1, D_0^1),$$

where $H_j^1 = (H_0^1, \ldots, H_r^1) \oplus 0_r$ and $H_j^1$ can be obtained by doing some difference transformation on $H_0^1$ for any $j = 1, \ldots, r$.

**Step 2:** Construct difference matrices $D_0^1, D_1^1, \ldots, D_r^1$ from the difference matrix $D(r(r+1), r(r+1); p)$ (obtained in Step 1).

The $D_0^1$ can be obtained by deleting $H_0^1$ from (15). Let

$$K^1 = (H_0^0 \oplus 0_r, D_0^1).$$

The $D_0^1$ can be changed into $D_j^1$ while $H_0^0 \oplus 0_r$ is changed to $H_j^1 \oplus 0_r$ by doing a difference transformation on $K^1$. The $D_j^1$ can be obtained by deleting $H_j^1 \oplus 0_r$ from the new construction of $K^1$. 


Step 3: Construct difference matrices \( D(r^m(r+1), r^m(r+1); p) \) by using the difference matrix \( D(r, r; r) \) and the difference matrices

\[
D_0^{m-1}, D_1^{m-1}, \ldots, D_r^{m-1}.
\]

Let \( H_0^m = D_0^{m-1} \oplus 0_r \) and \( H_j^m = (I_{m-2(r+1)} \otimes Q_j) \cdot (D_j^{m-1} \oplus 0_r) \), \( (j = 1, 2, \ldots, r) \), where \( Q_j's \) are \( r \times r \) permutation matrices satisfying

\[
L_{r^2}(r^{r+1}) = [(r) \oplus 0_r, Q_1 \cdot (r) \oplus 0_r, \ldots, Q_r \cdot (r) \oplus 0_r].
\]

In fact, the permutation matrices \( Q_1, \ldots, Q_r \) are obtained from the difference matrix \( D(r, r; r) \) since \( L_{r^2}(r^{r+1}) = (r) \oplus 0_r, D(r, r; r) \oplus (r) \) is an orthogonal array. By the relationship between the difference matrix \( D(r, r; r) = (d_{ij})_{r \times r} \) and permutation matrices \( MESC(d_{ij})_{r} \), we have (see (2))

\[
Q_j = \text{diag}(\sigma(d_{1j}), \ldots, \sigma(d_{rj})) K(r, r) = S_j K(r, r), \quad j = 1, 2, \ldots, r.
\]

Thus we obtain

\[
D(r^m(r+1), r^m(r+1); p) = (H_0^m, H_1^m, \ldots, H_r^m) = (H_0^m, D_m^m).
\]

Step 4: Construct difference matrices \( D_0^m, D_1^m, \ldots, D_r^m \) from \( D(r^m(r+1), r^m(r+1); p) \) in Step 3.

The \( D_0^m \) can be obtained by deleting \( H_0^m = D_0^{m-1} \oplus 0_r \) from the construction of \( D(r^m(r+1), r^m(r+1); p) \) in Step 3. Let

\[
K^m = (H_0^{m-1} \oplus 0_r, D_0^m),
\]

where \( H_0^{m-1} = D_0^{m-2} \oplus 0_r \). Then the \( D_0^m \) can be changed into \( D_j^m \) if \( H_0^{m-1} \oplus 0_r \) is changed into \( H_j^{m-1} \oplus 0_r \) by doing a difference transformation on \( K^m \), and the desired difference matrix \( D_j^m \) can be obtained by deleting \( H_j^{m-1} \oplus 0_r \) from the new construction of \( K^m \) for any \( m \geq 2 \).

Go to Step 3, the difference matrices \( D(r^m(r+1), r^m(r+1); p) \) for any \( m \geq 2 \) can be constructed.

5. Some examples

Now let us elaborate on the above steps through some examples. The particular structures of Hadamard matrices constructed in these examples are also interesting since the matrix image matrices of subarrays of the corresponding orthogonal arrays have clear and simple forms which can be obtained easily. The property can be used to construct the other new difference matrices and orthogonal arrays in our procedure.

Example 5.1. Let \( r = 2, \ p = 3 \).

The following difference \( D(r, r; r) \) and \( D(r+1, r+1; p) \) are known:

\[
D(2, 2; 2) = (d_{ij})_{2 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
D(3, 3; 3) = (d_{ij})_{3 \times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\]
over $G_2 = \{0, 1\}$, and
\[
D(3, 3; 3) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 2 \\
0 & 2 & 1 \\
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & A \\
\end{pmatrix} = (H_0^0, H_1^0, H_2^0)
\]
over $G = \{0, 1, 2\}$. By construction of Bose and Bush (1952), we have
\[
L_4(2^3) = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 1 & 2 \\
2 & 2 & 1 \\
\end{pmatrix} = [(2) \oplus 0_2, Q_1 \cdot (2) \oplus 0_2, Q_2 \cdot (2) \oplus 0_2],
\]
where
\[
Q_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} = K(2, 2),
\]
\[
Q_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\end{pmatrix} = \text{diag}(I_2, N_2)K(2, 2) = \text{diag}(1, N_3).
\]
Now let us construct the difference matrices $D(6, 6; 3)$ and $D(12, 12; 3), \ldots$.

**Step 1**: By Theorem 2.1, we have
\[
D(6, 6; 3) = \begin{pmatrix}
0 & A \oplus 0_2^T \\
A \oplus 0_2 & (Q_1(A \oplus 0_2), Q_2(A \oplus 0_2)) \\
\end{pmatrix},
\]
i.e.
\[
D(6, 6; 3) = \begin{pmatrix}
0 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 2 & 2 & 1 & 1 \\
1 & 2 & 1 & 2 & 1 & 2 \\
1 & 2 & 2 & 1 & 2 & 1 \\
2 & 1 & 1 & 2 & 2 & 1 \\
2 & 1 & 2 & 1 & 1 & 2 \\
\end{pmatrix} = (H_0^1, H_1^1, H_2^1) = (H_0^1, D_0^1),
\]
where $H_1^1$ (or $H_2^1$) can be obtained by adding $(1, 2, 0, 0, 0, 0)^T$ (or $(2, 1, 0, 0, 0, 0)^T$) to $H_2^1$ and further by timing the permutation matrix $\text{diag}(I_2, K(2, 2))$ (or $\text{diag}(1, N_3)$) on $H_0^1$. 
Step 2: Let

\[ K^1 = (H^0_0 \oplus 0_2, D^1_0) = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix}. \]

The \( D^1_1 \) (or \( D^1_2 \)) can be obtained by doing some difference transformations on \( K^1 \) such that \( H^0_0 \oplus 0_2 \) has been changed into \( H^0_1 \oplus 0_2 \) (or \( H^0_2 \oplus 0_2 \)). On the other word, we can get \((H^0_1 \oplus 0_2, D^1_1)\) (or \((H^0_2 \oplus 0_2, D^1_2))\) by adding \((0, 0, 1, 1, 2, 2)^T\) (or \((0, 0, 2, 2, 1, 1)^T\)) into \( K^1 \). By deleting \( H^0_1 \oplus 0_2 \) (or \( H^0_2 \oplus 0_2 \)) from \((H^0_1 \oplus 0_2, D^1_1)\) (or \((H^0_2 \oplus 0_2, D^1_2))\), we obtain \( D^1_1 \) (or \( D^1_2 \)) as follows:

\[ D^1_1 = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{or} \quad D^1_2 = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix}. \]

Step 3: Let

\[ H^2_0 = D^1_0 \oplus 0_2 \]

and

\[ H^2_j = I_3 \otimes Q_j \cdot D^1_j \oplus 0_2, \quad j = 1, 2. \]

Then we have

\[ D(12, 12; 3) = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 & 0 & 2 & 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 1 & 1 & 0 & 2 & 0 & 2 & 0 \\ 1 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 2 & 2 & 0 \\ 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 0 \\ 2 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 2 \end{pmatrix}. \]
Step 4: Let $D_0^2 = (H_1^2, H_2^2)$. Consider

$$K^2 = (H_0^1 \oplus 0_2, D_0^2) = (H_1^0, H_1^1, H_2^2).$$

The $D_1^2$ (or $D_2^2$) are obtained by doing the difference transformations on $K^2$ such that $H_0^1 \oplus 0_2$ has been changed into $H_1^1 \oplus 0_2$ (or $H_2^2 \oplus 0_2$). On the other word, we can get $H_1^1 \oplus 0_2, D_1^2$ (or $H_2^2 \oplus 0_2, D_2^2$) by adding $(1, 2, 0, 0, 0, 0)^T \oplus 0_2$ (or $(2, 1, 0, 0, 0, 0)^T \oplus 0_2$) into $K^2$, and further by timing diag($I_2, K(2, 2)) \otimes I_2$ (or diag($I_2, N_3 \otimes I_2$) onto the new $K^2$. By deleting $H_1^0 \oplus 0_2$ (or $H_0^0 \oplus 0_2$) from $(H_0^0 \oplus 0_2, D_1^1)$ (or $(H_0^0 \oplus 0_2, D_2^1)$), we can obtain $D_1^1$ (or $D_2^1$).

Similar to Step 3, define

$$H_j^2 = I_6 \otimes Q_j \cdot D_j^1 \oplus 0_2, \quad j = 0, 1, 2,$$

where $Q_0 = I_4$. Then we obtain the difference matrix $D(24, 24; 3)$ in Zhang (1991).

The other $D(2^m 3, 2^m 3; 3)$ can be constructed by doing the circles from Steps 3 to 4.

Note that the particular forms of the orthogonal arrays $L_{2^m 3}((2^m 3)^1 \cdot 3^m 3)$ is often used to construct the other new orthogonal arrays (or generalized Hadamard matrices) in the procedure of three steps or orthogonal-array addition (Zhang et al., 1999). In fact, the matrix images of submatrices of $L_{2^m 3}((2^m 3)^1 \cdot 3^m 3)$ can be easily obtained from above $D(2^m 3, 2^m 3; 3)$.

For example, by Step 1 and Corollary 3.1, we have $L_{18}(6^1 3^6) = (0_3 \oplus (6), c_1, \ldots, c_6)$ on which the matrix images of its submatrices $(c_1, c_2)$, $(c_3, c_4)$ and $(c_5, c_6)$ are $m(c_1, c_2) = \tau_3 \otimes \tau_3 \otimes P_2$, $m(c_3, c_4) = (M_1)(\tau_3 \otimes \tau_3 \otimes P_2)(M_1)^T$, $m(c_3, c_4) = (M_2)(\tau_3 \otimes \tau_3 \otimes P_2)(M_2)^T$, respectively, where (see Section 3.1)

$$M_1 = (K(3, 6)) \text{diag}(N_3, N_3^2, Q_1 \otimes I_3)(K(3, 6))^T,$$

$$M_2 = (K(3, 6)) \text{diag}(N_3^2, N_3, Q_2 \otimes I_3)(K(3, 6))^T.$$

The result in important because they can be used to construct other new orthogonal arrays such as $L_{36}(2^{10} 3^8 6^1)$ and $L_{36}(2^9 3^4 6^2)$ in Zhang et al. (2001), and can also be used to construct other new difference matrices such as $D(24, 20; 4)$ in Zhang et al. (2000).

Example 5.2. Let $r = 7$, $p = 4$. This is an example in which $r$ is a prime and $p \neq r + 1$.

Bose and Bush (1952) have given the generalized Hadamard matrix $D(7, 7; 7)$ as follows:

$$D(7, 7; 7) = (0(7), 2(7), \ldots, 6(7)) = (d_{ij})_{0 \leq i, j \leq 6} \mod 7$$

over the group $G_7 = \{0, 1, \ldots, 6\}$ where $d_{i0} = d_{0j} = 0$. By definition of modulus and letting $\sigma(x) = N_7^x \forall x \in G$, we have

$$\sigma(x)(7) = x + (7), \mod 7, \quad \forall x \in G.$$
On the other hand, Xiang (1983) has given

\[ D(8; 8; 4) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 3 & 0 & 1 & 2 \\
0 & 2 & 3 & 1 & 2 & 3 & 1 & 0 \\
0 & 3 & 1 & 2 & 1 & 3 & 0 & 2 \\
0 & 0 & 1 & 1 & 3 & 2 & 2 & 3 \\
0 & 1 & 3 & 2 & 0 & 2 & 3 & 1 \\
0 & 2 & 2 & 0 & 1 & 1 & 3 & 3 \\
0 & 3 & 0 & 3 & 2 & 1 & 2 & 1 \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & A \\
\end{bmatrix} = (0, D^{0}(8; 7; 4)) \]

over \( G_{4} = \{0, 1, 2, 3\} \) with the following table of addition whose Kronecker product \( \oplus \) (Shrikhande, 1964) is defined as follows:

\[ (0, 1, 2, 3)^{T} \oplus (0, 1, 2, 3) = \begin{bmatrix}
0 & 1 & 2 & 3 \\
1 & 0 & 3 & 2 \\
2 & 3 & 0 & 1 \\
3 & 2 & 1 & 0 \\
\end{bmatrix} . \]

By Theorem 2.1, we obtain

\[ D(56; 56; 4) = \begin{bmatrix}
0 & A \oplus 0^{T} \\
A \oplus 0^{T} (N_{7}^{0} A) \\
\end{bmatrix} . \]

By the recurrent construction steps in Theorem 3.4, difference matrices \( D(7^{m} 8; 7^{m} 8; 4) \), \( m \geq 2 \), can be constructed.

Note that the particular forms of orthogonal arrays \( L_{7^{m} 32}(7^{m} 8) \) can also be used to construct the other new orthogonal arrays since the matrix images of its submatrices can be obtained easily. Such as (see Section 3.1)

\[ T_{1}^{1} = K(4, 56) \text{diag}(\sigma_{4}(1), \sigma_{4}(2), \sigma_{4}(3), \sigma_{4}(0), \sigma_{4}(1), \sigma_{4}(2), \sigma_{4}(3)) \]

\[ [\text{diag}(I_{7}, N_{7}, \ldots, N_{7}^{3}) K(7, 7)] \oplus I_{4}) K(4, 56)^{T} , \]

where

\[ \sigma_{4}(0) = I_{4}, \quad \sigma_{4}(1) = I_{2} \oplus N_{2}, \quad \sigma_{4}(2) = N_{2} \oplus I_{2}, \quad \sigma_{4}(3) = N_{2} \oplus N_{2} \]

and \( L_{1} = T_{1}^{1} L_{0}^{1} \) in the following orthogonal array:

\[ L_{224}(56)^{4 56} = (0_{4} \oplus (56), (4) \oplus D(56, 56; 4)) =: (0_{4} \oplus (56), L_{0}^{1}, L_{1}^{1}, \ldots, L_{7}^{1}) , \]

i.e., \( L_{0}^{1} = (4) \oplus D^{0}(8; 7; 4) \oplus 0_{7} \) and \( L_{1}^{1} = T_{1}^{1} [L_{0}^{1}] \). On the other hand, we have

\[ m(L_{0}^{1}) = \tau_{4} \otimes \tau_{8} \otimes P_{7} \quad \text{and} \quad m(L_{1}^{1}) = T_{1}^{1} (\tau_{4} \otimes \tau_{8} \otimes P_{7})(T_{1}^{1})^{T} . \]
Example 5.3. Let \( r = 9 \), \( p = 5 \). This is an example where \( r \) is not prime and \( p \neq r + 1 \).

Xiang (1983) has given a generalized Hadamard matrix \( D(9,9;9) \) as follows:

\[
D(9,9;9) = (d_{ij})_{9\times 9} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \\
0 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \\
0 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 \\
0 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
0 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 \\
0 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{pmatrix}
\]

over \( G_9 = \{0,1,\ldots,8\} \) whose additive table is the following array:

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 5 & 6 & 4 & 6 & 0 & 3 & 2 & 7 \\
2 & 8 & 6 & 1 & 5 & 7 & 6 & 4 & 3 \\
3 & 4 & 1 & 7 & 2 & 6 & 8 & 0 & 5 \\
4 & 6 & 5 & 2 & 8 & 3 & 7 & 1 & 0 \\
5 & 0 & 7 & 6 & 3 & 1 & 4 & 8 & 2 \\
6 & 3 & 0 & 8 & 7 & 4 & 2 & 5 & 1 \\
7 & 2 & 4 & 0 & 1 & 8 & 5 & 3 & 6 \\
8 & 7 & 3 & 5 & 0 & 2 & 1 & 0 & 4
\end{pmatrix} = [\sigma(0)(9),\sigma(1)(9),\ldots,\sigma(8)(9)],
\]

where \( \sigma(j) \) is a permutation matrix such that \( \sigma(j)(9) = j + (9) \) for any \( j \in G_9 = \{0,1,\ldots,8\} \).

On the other hand, a generalized Hadamard matrix \( D(10,10;5) \), due to Xu (1979), is given as follows:

\[
D(10,10;5) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\
0 & 2 & 4 & 1 & 3 & 2 & 4 & 1 & 3 & 0 \\
0 & 3 & 1 & 4 & 2 & 1 & 4 & 2 & 0 & 3 \\
0 & 4 & 3 & 2 & 1 & 2 & 1 & 0 & 4 & 3 \\
0 & 0 & 4 & 2 & 4 & 1 & 3 & 3 & 1 & 2 \\
0 & 1 & 1 & 0 & 3 & 3 & 2 & 4 & 4 & 2 \\
0 & 2 & 3 & 3 & 2 & 4 & 0 & 4 & 1 & 1 \\
0 & 3 & 0 & 1 & 1 & 4 & 3 & 3 & 2 & 4 \\
0 & 4 & 2 & 4 & 0 & 3 & 2 & 1 & 2 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & A
\end{pmatrix}
\]

over \( G_5 = \{0,1,2,3,4\} \) with the usual addition of modulus 5.
By Theorem 2.1, we obtain
\[
D(90, 90; 5) = \begin{pmatrix}
0 & A \oplus 0^T \\
A \oplus 0 & (\sigma(d_{ij})A)
\end{pmatrix}.
\]
By the recurrent construction steps in Theorem 3.4, \(D(9^m10, 9^m10; 5)\) \((m \geq 2)\) can be constructed.

Also note that the particular forms of orthogonal arrays \(L_{9^m50}((9^m10)^15^m10)\) can also be used to construct the other new orthogonal arrays since the matrix images of its submatrices can be obtained easily. Such as (see Section 3.1)
\[
T_1^1 = K(5, 90)\text{diag}(N_5, N_5^2, N_5^3, I_5, N_5, N_5^2, N_5^3),
\]
\[
[\text{diag}(\sigma(0), \sigma(1), \ldots, \sigma(8))K(9, 9)] \oplus I_5)K(5, 90)^T
\]
where \(L_1^1 = T_1^1L_0^1\) in the orthogonal array
\[
L_{450}(90^15^90) = (0_5 \oplus (90), (5) \oplus D(90, 90; 5)) =: (0_5 \oplus (90), L_0^1, L_1^1, \ldots, L_9^1).
\]

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